PERGAMON

International Journal of Solids and Structures 37 (2000) 3201–3229

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The elastic and electric fields for three-dimensional contact for transversely isotropic piezoelectric materials

Hao-Jiang Ding*, Peng-Fei Hou, Feng-Lin Guo

Department of Civil Engineering, Zhejiang University, Hangzhou, 310027, People's Republic of China

Received 30 May 1998; in revised form 14 January 1999

Abstract

Firstly, the extended Boussinesq and Cerruti solutions for point forces and point charge acting on the surface of a transversely isotropic piezoelectric half-space are derived. Secondly, aiming at a series of common three-dimensional contact including spherical contact, a conical indentor and an upright circular flat punch on a transversely isotropic piezoelectric half-space, we solve for their elastic and electric fields in smooth and frictional cases by first evaluating the displacement functions and then differentiating. The displacement functions can be obtained by integrating the extended Boussinesq or Cerruti solutions in the contact region. Then, when only normal pressure is loaded, the stresses in the half-space of PZT-4 piezoelectric ceramic are compared in the figures with those of the transversely isotropic material which are assumed to have the same elastic constants as those of PZT-4. Meanwhile, the electric components in the half-space of PZT-4 piezoelectric ceramic are also shown in the same figures. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Since Hertz (1882) published his classic article 'On the contact of elastic solids', the research on the contact of elastic materials has been conducted for more than one hundred years, and a lot of scientists including mathematicians, physicists and civil engineers all contributed to this area. Earlier research concentrated on isotropic materials, the corresponding methods were efficient and the analytic solutions in many cases were obtained. The authors want to mention the method of complex functions put forward by Muskhelishvili (1953) and integral transform initiated by Sneddon (1951). These two methods are widely used not only in contact problems but also in other problems of elastic mechanics, and they are still effective for a lot of modern research on

* Tel: +0086-571-799-3057; Fax: +0086-571-795-1012.

PII: S0020-7683(99)00027-X

classic contact problems. In addition, it was Mindlin (1949) who first studied the bonded and slip contact of isotropic materials and made the research on contact problems more comprehensive. Gladwell (1980) and Johnson (1985) systematically reviewed the history and literatures of the contact problems of isotropic elastic materials.

With the development of modern science and technology, many kinds of anisotropic materials with advantageous characteristics are increasingly used. For example, high-strength roadbeds and metal surface treating layer, etc., are all anisotropic to a certain degree, so the anisotropic effect must be considered when we study the contact of these kinds of elastic materials. Elliott (1948, 1949) first consider the axisymmetric contact of a rigid sphere, a conical indentor and a cylinder punch on a transversely isotropic half-space respectively. Shield (1951) considered the contact of an elliptic flat punch. Conway et al. (1967), Conway and Farnham (1967) considered the sliding rigid sphere acting on a transversely isotropic half-space for stress on the axis of symmetry. Using Fourier transforms, Willis (1966, 1967) transformed the Hertzian contact problem of anisotropic materials into evaluating contour integrals and some explicit formulae on the surface of half-space were presented for transversely isotropic media. Chen (1969) made further research on the plane contact of anisotropic materials and three-dimensional contact of transversely isotropic materials, fully considering the stress distribution for indentation and sliding. He brought out that in certain important practical situations of plane contact the stress functions for the isotropic and anisotropic materials are of the same form. He also found that the boundary value problem of sphere contact of transversely isotropic materials is the same as that of isotropic materials and accordingly obtained the expressions of displacement and stress components in the former case by virtue of the isotropic result of Hamilton and Goodman (1966). Keer and Mowry (1979) extended Mindlin's (1949) research on the bonded and slip contact of isotropic materials to the case of transversely isotropic materials. Lin et al. (1991) considered Hertzian and non-Hertzian contact using the three potential functions which had been used by Green and Zerna (1954), Pan and Chou (1976). Since the seventies, Fabrikant (1989, 1991) put forward some new methods in the potential theory and solved a great variety of mathematical and physical problems. Especially in contact problems, the solutions of corresponding boundary value problems could be expressed in integral forms and even be given in the form of elementary functions in some cases. Thereby, complicated transforms are avoided. By this means, Hanson (1992a, 1992b, 1994) fully studied various kinds of indentors on a transversely isotropic half-space with the effect of sliding friction included, and obtained analytic expressions for components of displacement and stress in the half-space.

Since the Curie brothers found the piezoelectric effect (Mason, 1981) in 1880, due to its characteristic direct-converse piezoelectric effect, the piezoelectric materials are used more and more in manufacturing various sensors and actuators, and have important applications (Pohanka and Smith, 1988), in such hi-tech areas as electronics, laser, supersonics, microwave, infrared, navigation, biology, etc. Piezoelectric ceramic is a kind of transversely isotropic piezoelectric material and has found widespread applications because of its excellent piezoelectricity. But because of the intrinsic brittleness of piezoelectric ceramic, the stress concentration, near the contact region and caused by inharmonious contact between the components of piezoelectric ceramic and the other components, could cause the piezoelectric component failure. Therefore, it is necessary to make theoretical analysis and accurate quantitative description of the elastic and electric fields for contact of piezoelectric ceramic, from the point of view of mechanical-electric coupling. Sosa and Castro (1994) obtained the analytic solution for point force and point charge acting on an orthotropic piezoelectric half-plane through a state space methodology in conjunction with the Fourier transforms. Wang and Zheng (1995) gave the analytic solution for a concentrated lateral shear force acting on a transversely isotropic piezoelectric half-space. Ding et al. (1996) gave the solution of point forces and point charge acting on the boundary of a piezoelectric half-space. All above solutions were derived using transform methods. Since the final expressions of these solutions are relatively tedious and complex, it is of little practical significance. In addition, Fan et al. (1996) studied the two-dimensional contact on a piezoelectric half-plane by using Stroh's formalism, and gave the solutions for loads acting on the boundary of an anisotropic piezoelectric half-plane.

In the present paper, the extended Boussinesq and Cerruti solutions for point forces and point charge acting on the surface of a transversely isotropic piezoelectric half-space are first derived. Secondly, aiming at a series of three-dimensional contact problems including spherical contact, a conical indentor and an upright circular flat punch on a transversely isotropic piezoelectric half-space, which arise often in engineering applications, we solve for their elastic and electric fields in the smooth and frictional cases by first evaluating the displacement functions and then differentiating. The displacement functions can be obtained by integrating the extended Boussinesq or Cerruti solutions in the contact region. Finally, when only a normal pressure is applied, the stresses in the half-space of PZT-4 piezoelectric ceramic are contrasted in the figures with those of the transversely isotropic material which is assumed to have the same elastic constants as those of PZT-4. Additionally, the electric components in the half-space of PZT-4 piezoelectric ceramic are also given in the same figures.

2. General solution for transversely isotropic piezoelectric media

We introduce Cartesian coordinates $Oxyz$ with the z-axis perpendicular to planes of isotropy. In the case of characteristic distinct roots, $s_1 \neq s_2 \neq s_3 \neq s_1$, Ding et al. (1996) gave the general solutions for the displacement and electric potential in terms of four displacement functions for transversely isotropic piezoelectric media, so that the displacement functions ψ_j satisfy, respectively, the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_j^2}\right)\psi_j = 0, \ (j = 0,1,2,3) \tag{1}$$

where $z_j = s_j z$ (j = 0,1,2,3) and $s_0 = \sqrt{c_{66}/c_{44}}$,$s_j$ ($j = 1,2,3$) are the three characteristic roots of a sixth degree equation defined in Ding et al. (1996) and satisfy $\mathrm{Re}(s_j) > 0$.

Using the constitutive relation, the general solutions for the stress and the electric displacement expressed by four displacement functions are obtained. At this point, the coefficients in front of derivatives of displacement functions with respect to coordinates are all products or linear combinations of material constants and characteristic roots. If expressions for the stresses and electric displacements are substituted into the equilibrium and Gauss equations, some relations among these coefficients will be determined through consideration of Eq. (1). With these relations taken into account, the general solutions for stress and electric displacement can be obtained.

For the sake of convenience, the following notations are introduced:

$$U = u + iv, \ w_1 = w, w_2 = \Phi, \ \sigma_1 = \sigma_x + \sigma_y$$

$$\sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy}, \ \sigma_{z1} = \sigma_z, \ \sigma_{z2} = D_z$$

$$\tau_{z1} = \tau_{xz} + i\tau_{yz}, \ \tau_{z2} = D_x + iD_y \tag{2}$$

Then the general solution can be concisely written as follows:

$$U = \Lambda\left(i\psi_0 + \sum_{j=1}^{3}\psi_j\right), \ w_m = \sum_{j=1}^{3} s_j k_{mj}\frac{\partial\psi_j}{\partial z_j}$$

$$\sigma_1 = 2\sum_{j=1}^{3}(m_j - c_{66})\frac{\partial^2\psi_j}{\partial z_j^2},\ \sigma_2 = 2c_{66}\Delta^2\left(i\psi_0 + \sum_{j=1}^{3}\psi_j\right)$$

$$\sigma_{zm} = \sum_{j=1}^{3}\omega_{mj}\frac{\partial^2\psi_j}{\partial z_j^2},\ \tau_{zm} = \Delta\left(s_0\rho_m i\frac{\partial\psi_0}{\partial z_0} + \sum_{j=1}^{3}s_j\omega_{mj}\frac{\partial\psi_j}{\partial z_j}\right),\ (m = 1,2) \tag{3}$$

where k_{mj} are constants dependent on material constants and characteristic roots, and

$$\omega_{1j} = c_{44}(1 + k_{1j}) + e_{15}k_{2j},$$

$$\omega_{2j} = e_{15}(1 + k_{1j}) - \varepsilon_{11}k_{2j}$$

$$m_j = 2c_{66} - \omega_{1j}s_j^2,\ \rho_1 = c_{44},\ \rho_2 = e_{15},$$

$$\Delta = \partial/\partial x + i\partial/\partial y,\ (j = 1,2,3) \tag{4}$$

3. The solutions for forces and charge acting on a transversely isotropic piezoelectric half-space

Considering a transversely isotropic piezoelectric half-space $z \geq 0$ with the surface $z = 0$ parallel to the planes of isotropy, the extended Boussinesq and Cerruti solutions for point forces and point charge acting on the surface of half-space are first derived. Then, the displacement w on surface, caused by a hemiellipsoidal normal pressure and electric displacement acting on the surface is given by using the Superposition principle.

3.1. The extended Boussinesq solution for a normal point force P_0 and a point charge Q_0 acting on the surface

When a normal point force P_0 and a point charge Q_0 act on the coordinate origin, this is an axisymmetric problem. We can assume the functions ψ_0 and ψ_{j} in the following form:

$$\psi_0 = 0,\ \psi_j = A_j\ln R_j^*,\ (j = 1,2,3) \tag{5}$$

where $R_j^* = R_j + z_j, R_j = \sqrt{r^2 + z_j^2}$ and $r^2 = x^2 + y^2$, A_j are undetermined constants.

Substituting Eq. (5) into Eq. (3), we have

$$U = \sum_{j=1}^{3}\frac{A_j x}{R_j R_j^*} + i\sum_{j=1}^{3}\frac{A_j y}{R_j R_j^*},\ w_m = \sum_{j=1}^{3}\frac{A_j s_j k_{mj}}{R_j}$$

$$\sigma_1 = 2c_{66}\sum_{j=1}^{3}A_j\left[\frac{2}{R_j R_j^*} - \frac{x^2 + y^2}{R_j^3 R_j^*} - \frac{x^2 + y^2}{R_j^2 R_j^{*2}}\right] - 2\sum_{j=1}^{3}m_j\frac{A_j z_j}{R_j^*}$$

$$\sigma_2 = -2c_{66}\sum_{j=1}^{3}A_j(x^2-y^2)\left(\frac{1}{R_j^3R_j^*}+\frac{1}{R_j^2R_j^{*2}}\right) - i2c_{66}\sum_{j=1}^{3}A_jxy\left(\frac{1}{R_j^3R_j^*}+\frac{1}{R_j^2R_j^{*2}}\right)$$

$$\sigma_{zm} = -\sum_{j=1}^{3}\omega_{mj}\frac{A_jz_j}{R_j^3}, \tau_{zm} = -\sum_{j=1}^{3}\omega_{mj}\frac{A_js_jx}{R_j^3} - i\sum_{j=1}^{3}\omega_{mj}\frac{A_js_jy}{R_j^3} \tag{6}$$

The boundary conditions at $z = 0$ require

$$\tau_{zl} = 0, \ \sigma_{zm} = 0, \ (m = 1,2) \tag{7}$$

Obviously, both conditions $\sigma_{zm}=0$ are satisfied automatically. Substituting Eq. (6) into $\tau_{zl}=0$, we have

$$\sum_{j=1}^{3}s_j\omega_{1j}A_j = 0 \tag{8}$$

Meanwhile, taking into consideration all the equilibrium conditions, apart from those already satisfied, for the layer cut from the infinite piezoelectric half-space by the two planes $z = 0$ and $z=h$ we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma_{zm}(x,y,h)dxdy + P_m = 0, \ (m = 1,2) \tag{9}$$

where

$$P_1 = P_0, P_2 = -Q_0 \tag{10}$$

Substituting σ_{zm} of Eq. (6) into Eq. (9), we have

$$2\pi\sum_{j=1}^{3}\omega_{mj}A_j = P_m, \ (m = 1,2) \tag{11}$$

Combining Eq. (8) and Eq. (11) to determine A_j, we obtain

$$A_j = \delta_jP_0 + \lambda_jQ_0 \tag{12}$$

where

$$\delta_1 = (s_2\omega_{12}\omega_{23} - s_3\omega_{13}\omega_{22})/\Delta_1, \ \lambda_1 = \omega_{12}\omega_{13}(s_2 - s_3)/\Delta_1$$

$$\delta_2 = (s_3\omega_{13}\omega_{21} - s_1\omega_{11}\omega_{23})/\Delta_1, \ \lambda_2 = \omega_{11}\omega_{13}(s_3 - s_1)/\Delta_1$$

$$\delta_3 = (s_1\omega_{11}\omega_{22} - s_2\omega_{12}\omega_{21})/\Delta_1, \ \lambda_3 = \omega_{11}\omega_{12}(s_1 - s_2)/\Delta_1 \tag{13}$$

and

$$\Delta_1 = 2\pi\begin{vmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{11} & \omega_{12} & \omega_{13} \end{vmatrix} \tag{14}$$

Substituting Eq. (12) into Eq. (8) and Eq. (11), because P_0 and Q_0 can be arbitrary value, we can get the following equations:

$$\sum_{j=1}^{3} s_j\omega_{1j}\delta_j = 0, \sum_{j=1}^{3} s_j\omega_{1j}\lambda_j = 0,\ 2\pi\sum_{j=1}^{3}\omega_{1j}\delta_j = 1$$

$$\sum_{j=1}^{3}\omega_{1j}\lambda_j = 0, \sum_{j=1}^{3}\omega_{2j}\delta_j = 0,\ 2\pi\sum_{j=1}^{3}\omega_{2j}\lambda_j = -1 \tag{15}$$

In order to study the contact problems of piezoelectric materials, the displacement w at a point on the surface, with a distance of r from the origin, is given as follows.

$$w = \sum_{j=1}^{3}\frac{A_j s_j k_{1j}}{R_j} = \frac{KP_0 + LQ_0}{r} \tag{16}$$

where

$$K = \sum_{j=1}^{3} s_j k_{1j}\delta_j, L = \sum_{j=1}^{3} s_j k_{1j}\lambda_j \tag{17}$$

Eq. (16) shows that the displacement w on the surface is in inverse proportion to r.

3.2. The extended Cerruti solution for tangential point forces P_x and P_y acting on the surface

When only P_x acts on the coordinate origin in the positive x direction, we assume

$$\psi_0 = \frac{B_0 y}{R_0^*},\ \psi_j\frac{B_j x}{R_j^*},\ (j = 1,2,3) \tag{18}$$

where B_0 and B_j are undetermined constants.

Substituting Eq. (18) into Eq. (3), we have

$$U = -B_0\left(\frac{1}{R_0^*} - \frac{y^2}{R_0 R_0^{*2}}\right) + \sum_{j=1}^{3} B_j\left(\frac{1}{R_j^*} - \frac{x^2}{R_j R_j^{*2}}\right) - i\left(B_0\frac{xy}{R_0 R_0^{*2}} + \sum_{j=1}^{3} B_j\frac{xy}{R_j R_j^{*2}}\right)$$

$$w_m = -\sum_{j=1}^{3} s_j k_{mj} B_j \frac{x}{R_j R_j^*}$$

$$\sigma_1 = -2c_{66}\sum_{j=1}^{3} B_j\left[\frac{4x}{R_j R_j^{*2}} - \frac{x^3 + xy^2}{R_j^3 R_j^{*2}} - \frac{2(x^3 + xy^2)}{R_j^2 R_j^{*3}}\right] + 2\sum_{j=1}^{3} m_j B_j\left(\frac{xz_j}{R_j^3 R_j^*} + \frac{x}{R_j^2 R_j^*}\right)$$

$$\sigma_2 = 4c_{66}B_0\left(\frac{x}{R_0 R_0^{*2}} - \frac{xy^2}{R_0^3 R_0^{*2}} - \frac{2xy^2}{R_0^2 R_0^{*3}}\right) - 2c_{66}\sum_{j=1}^{3} B_j\left[\frac{2x}{R_j R_j^{*2}} - \frac{x^3 - xy^2}{R_j^3 R_j^{*2}} - \frac{2(x^3 - xy^2)}{R_j^2 R_j^{*3}}\right]$$

$$+i\left\{c_{66}B_0\left[\frac{2y}{R_0R_0^{*2}}+\frac{y(x^2-y^2)}{R_0^3R_0^{*2}}+\frac{2y(x^2-y^2)}{R_0^2R_0^{*3}}\right]\right.$$

$$\left.+2c_{66}\sum_{j=1}^{3}B_j\left(-\frac{y}{R_jR_j^{*2}}+\frac{x^2y}{R_j^3R_j^{*2}}+\frac{2x^2y}{R_j^2R_j^{*3}}\right)\right\}$$

$$\sigma_{zm}=\sum_{j=1}^{3}\omega_{mj}B_j\left(\frac{xz_j}{R_j^3R_j^*}+\frac{x}{R_j^2R_j^*}\right)$$

$$\tau_{zm}=s_0\rho_mB_0\left(\frac{1}{R_0R_0^*}-\frac{y^2z_0}{R_0^3R_0^{*2}}-\frac{2y^2}{R_0^2R_0^{*2}}\right)-\sum_{j=1}^{3}s_j\omega_{mj}B_j\left(\frac{1}{R_jR_j^*}-\frac{x^2z_j}{R_j^3R_j^{*2}}-\frac{2x^2}{R_j^2R_j^{*2}}\right)$$

$$+i\left[s_0\rho_mB_0\left(\frac{xyz_0}{R_0^3R_0^{*2}}+\frac{2xy}{R_0^2R_0^{*2}}\right)+\sum_{j=1}^{3}s_j\omega_{mj}B_j\left(\frac{xyz_j}{R_j^3R_j^{*2}}+\frac{2xy}{R_j^2R_j^{*2}}\right)\right] \quad (19)$$

Substituting Eq. (19) into boundary conditions (7), we have

$$s_0c_{44}B_0+\sum_{j=1}^{3}s_j\omega_{1j}B_j=0 \quad (20)$$

$$\sum_{j=1}^{3}\omega_{mj}B_j=0,\ (m=1,2) \quad (21)$$

Meanwhile, taking into consideration all the equilibrium conditions for the layer cut from the infinite piezoelectric half-space by two planes $z = 0$ and $z = h$, we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tau_{zx}(x,y,h)dxdy+P_x=0 \quad (22)$$

Substituting τ_{zx} in Eq. (19) into Eq. (22), we have

$$s_0c_{44}B_0-\sum_{j=1}^{3}s_j\omega_{1j}B_j=-\frac{P_x}{\pi} \quad (23)$$

Then combining Eq. (20) and Eq. (21) with Eq. (23) to determine B_j, we have

$$B_j=P_x\eta_j,\ (j=0,1,2,3) \quad (24)$$

where

$$\eta_0=-1/(2\pi s_0c_{44}),$$

$$\eta_1=(\omega_{12}\omega_{23}-\omega_{13}\omega_{22})/\Delta_2$$

$$\eta_2 = (\omega_{13}\omega_{21} - \omega_{11}\omega_{23})/\Delta_2,$$

$$\eta_3 = (\omega_{11}\omega_{22} - \omega_{12}\omega_{21})/\Delta_2 \tag{25}$$

and

$$\Delta_2 = 2\pi \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} \end{vmatrix} \tag{26}$$

When only P_y acts on the coordinate origin in the positive y direction, we can obtain the displacement functions ψ_0 and ψ_j by means of changing x, y and P_x in Eq. (18) into y, $-x$ and P_y, respectively. In the same way as above, we can obtain B_j in the same form as Eq. (24) by replacing P_x with P_y. Therefore, when P_x and P_y act simultaneously, the displacement functions ψ_0 and ψ_j are

$$\psi_0 = \eta_0\left(P_x\frac{y}{R_0^*} - P_y\frac{x}{R_0^*}\right),\ \psi_j = \eta_j\left(P_x\frac{x}{R_j^*} + P_y\frac{y}{R_j^*}\right) \tag{27}$$

3.3. The displacement functions for point forces and point charge acting on an arbitrary point on the surface

When cylindrical coordinates (r, θ, z) is adopted. The point charge Q_0 and three point forces P_x, P_y and P_0 in the positive x, y and z direction act on an arbitrary point $M(r_0,\theta_0,0)$ on the surface of a transversely isotropic piezoelectric half-space.

According to Eq. (5), Eq. (12) and Eq. (27), we obtain the following displacement functions by using a shift of origin

$$\psi_0(r,\theta,z;r_0,\theta_0) = iG_0(P\bar{\Delta} - \bar{P}\Delta)\chi(z_0)$$

$$\psi_j(r,\theta,z,r_0,\theta_0) = G_j(P\bar{\Delta} + \bar{P}\Delta)\chi(z_j) + (P_0\delta_j + Q_0\lambda_j)R_j^*,\ (j = 0,1,2,3) \tag{28}$$

where

$$R_j^* = R_j + z_j, R_j = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0) + z_j^2}$$

$$\chi(z_j) = z_j\ln R_j^* - R_j, G_j = -\eta_j/2 \tag{29}$$

constants δ_j and λ_j are expressed in Eq. (13), η_j is expressed in Eq. (25), $P = P_x + iP_y$ is a complex shear force $\bar{P}$ and $\bar{\Delta}$ are the complex conjugate of P and Δ, respectively.

3.4. The displacement w on the surface of half-space caused by hemiellipsoidal normal pressure and electric displacement acting on the surface

Assume there are hemiellipsoidal normal pressure $p(x,y)$ and electric displacement $d(x,y)$ acting on the surface, namely

$$p(x,y) = p_0\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}$$

$$d(x,y) = d_0\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} \text{ Within } S: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{30}$$

where p_0 and d_0 are the pressure and electric displacement at the center of the ellipse S, a and b are the semi-axes of the ellipse, $p(x,y)$ and $d(x,y)$ have the same positive direction as the z-axis.

Using superposition principle and Eq. (16), we can obtain the displacement w on the surface as

$$w(x,y) = K\int\int_S \frac{p(\xi,\eta)}{r}d\xi d\eta + L\int\int_S \frac{d(\xi,\eta)}{r}d\xi d\eta \tag{31}$$

where $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$.

Substituting Eq. (30) into Eq. (31), we have

$$w = \frac{\pi(Kp_0 + Ld_0)}{a}\left\{abK(e) - \frac{b}{a}D(e)x^2 - \frac{a}{b}[K(e) - D(e)]y^2\right\} \tag{32}$$

where $e = \sqrt{1 - (b/a)^2}$ is the eccentricity of the ellipse S, and

$$K(e) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - e^2\sin^2\varphi}}$$

$$E(e) = \int_0^{\pi/2} \sqrt{1 - e^2\sin^2\varphi}d\varphi$$

$$D(e) = [K(e) - E(e)]/e^2 \tag{33}$$

In the case of $e = 0$ (the ellipse reduces to a circle), we have $K(0) = E(0) = \pi/2$, $D(0) = \pi/4$ and

$$w = \frac{\pi^2(Kp_0 + Ld_0)}{4a}(2a^2 - x^2 - y^2) \tag{34}$$

4. The contact region and contact loads for contact between a piezoelectric body and another body under forces and charges

Consider that a body 1 (which may or may not be piezoelectric) and a piezoelectric body 2 touch at the point O before the forces and charges act on them. Regarding O as the origin, we introduce two Cartesian coordinates $Ox_1y_1z_1$ and $Ox_2y_2z_2$, where the z_1 and z_2 axes coincide with the common normal line with their positive directions pointing into body 1 and body 2, respectively. x_1, y_1 axes and x_2, y_2 axes are all in the common tangent plane and also in the principal plane of curvature of bodies 1 and 2, respectively. All the above are shown in Fig. 1.

The surface equations of bodies 1 and 2 may be written as

$$z_1 = F_1(x_1,y_1),\ z_2 = F_2(x_2,y_2) \tag{35}$$

Expanding $F_1(x_1,y_1)$ and $F_2(x_2,y_2)$ into Taylor's series around the origin O and only retaining the quadratic terms, we have

$$z_1 = (K_{11}x_1^2 + K_{12}y_1^2)/2$$

$$z_2 = (K_{21}x_2^2 + K_{22}y_2^2)/2 \tag{36}$$

where K_{11}, K_{12} and K_{21}, K_{22} are the principal curvatures of bodies 1 and 2 at the point O, respectively. Generally, the x_1 and x_2 axes do not coincide and the angle between them is assumed to be ω. We introduce a common coordinate system $Oxyz$ where $z = z_2$, and assume the angles between the y-axis and the x_1 and x_2 axes are ω_1 and ω_2 respectively, as shown in Fig. 2. Then, the surface Eq. (36) can be expressed in the common coordinates x and y by using the coordinate transform formalism.

The distance between M_1 and M_2, which have the same coordinate (x,y), could be expressed as $\overline{M_1M_2} = z_1 + z_2$. After diagonalization of the quadratic form for $z_1 + z_2$, we obtain

$$z_1 + z_2 = Ax^2 + By^2 \tag{37}$$

where A and B should be calculated as follows:

$$2A = K_{11}\sin^2\omega_1 + k_{12}\cos^2\omega_1 + K_{21}\sin^2\omega_2 + K_{22}\cos^2\omega_2$$

$$2B = K_{11}\cos^2\omega_1 + k_{12}\sin^2\omega_1 + K_{21}\cos^2\omega_2 + K_{22}\sin^2\omega_2 \tag{38}$$

Because $z_1 + z_2$ is positive, the constants A and B are positive and can be obtained from Eq. (38).

We further consider that body 1 and body 2 are pressed to each other by a pair of forces P_z. Meanwhile, a pair of charges $+Q$ and $-Q$ locate at two points on the common normal line and in body

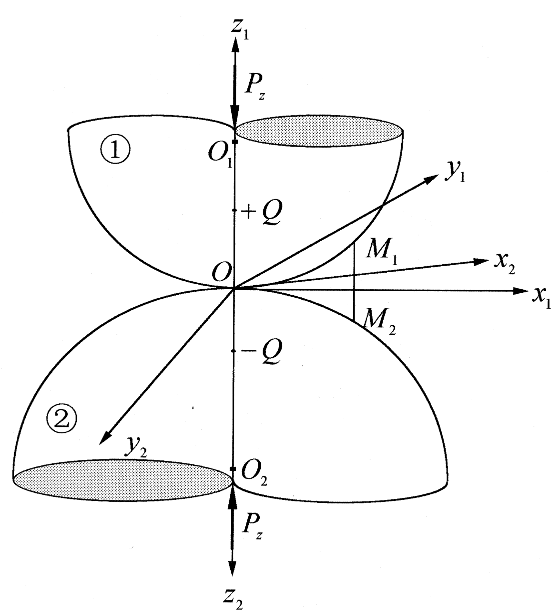

Fig. 1. Geometry of two piezoelectric bodies touched at the point O.

1 and body 2, respectively. The forces and charges are distant from the contact point O. Because of deformation, a contact region forms around the initial contact point O. Obviously, the dimensions of the contact region, the values of the normal pressure $p(x,y)$ and electric displacement $d(x,y)$ inside the contact region are closely related to the values of P_z and Q. It should be noted that the deformation just takes place near the contact region, and there is no deformation in the place such as points O_1 and O_2, which are distant from the contact point O.

Because of this deformation, points O_1 and O_2 approach to each other by a certain amount δ. This relative displacement δ is a rigid body displacement, so points M_1 and M_2 also to approach each other by the same amount. In addition, M_1 gets a displacement of $\tilde{w}_1$ in the positive z_1 direction and M_2 gets a displacement of $\tilde{w}_2$ in the positive z_2 direction. The distance between M_1 and M_2 will diminish totally by $\delta - \tilde{w}_1 - \tilde{w}_2$.

Obviously, if

$$z_1 + z_2 = \delta - (\tilde{w}_1 + \tilde{w}_2) \tag{39}$$

then M_1 and M_2 will contact each other. Otherwise, if

$$z_1 + z_2 > \delta - (\tilde{w}_1 + \tilde{w}_2) \tag{40}$$

then M_1 and M_2 will be still separated and out of the contact region.

Assume that

1. The shape of contact region S is elliptical and its dimensions are sufficiently small compared with those of the bodies 1 and 2, so we can regard them as two half-spaces;
2. There are no friction and free charge in the contact region and we call such contact 'smooth';

In addition, because the dielectric constant, or permittivity, of piezoelectric ceramics is 10^3 times

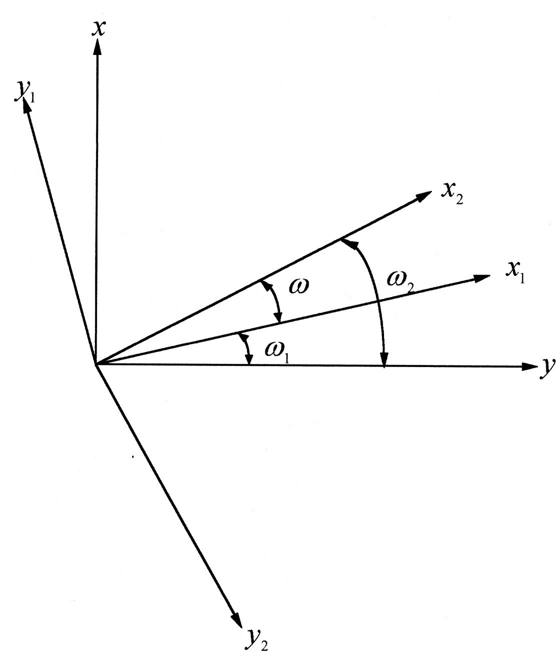

Fig. 2. Common coordinate system for the contact of two piezoelectric bodies.

higher than the environment (e.g. air), we further assume that

3. The normal electric displacement on the surface of bodies 1 and 2 is nonzero only inside the contact region S. The contact pressure $p(x,y)$ and electric displacement $d(x,y)$ inside the contact region distribute in the form of Eq. (30).

Therefore, when the common tangential plane is parallel to the planes of isotropy, we can obtain $\tilde{w}_1$ and $\tilde{w}_2$ using Eq. (31), in the form

$$\tilde{w}_1 = K_1 \int\int_S \frac{p(\xi,\eta)}{r} d\xi d\eta - L_1 \int\int_S \frac{d(\xi,\eta)}{r} d\xi d\eta$$

$$\tilde{w}_2 = K_2 \int\int_S \frac{p(\xi,\eta)}{r} d\xi d\eta + L_2 \int\int_S \frac{d(\xi,\eta)}{r} d\xi d\eta \tag{41}$$

where $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$.

For piezoelectric bodies, K_n and L_n can be obtained from Eq. (17) as follows:

$$K_n = \left(\sum_{j=1}^{3} s_j k_{1j} \delta_j\right)_n, \; L_n = \left(\sum_{j=1}^{3} s_j k_{1j} \lambda_j\right)_n \tag{42}$$

where subscripts $n = 1,2$ correspond to bodies 1 and 2; when body 1 is transversely isotropic medium (the elastic field is uncoupled from electric field), $L_1 = 0$, and according to Ding et al. (1997), we have

$$K_1 = \frac{(s_1 + s_2)c_{11}}{2\pi s_1 s_2 (c_{11}c_{33} - c_{13}^2)} \tag{43}$$

where c_{ij} are elastic constants, $s_k (k = 1,2)$ are two characteristic roots of a fourth degree equation defined in Hu (1953) and satisfy $\mathrm{Re}(s_k) > 0$; when body 1 is rigid, we take $L_1 = 0$, $K_1 = 0$.

Substituting Eq. (37) and Eq. (41) into Eq. (39), we have

$$\delta = (Ax^2 + By^2) = \frac{c_p}{\pi} \int\int_S \frac{p(\xi,\eta)}{r} d\xi d\eta + \frac{c_d}{\pi} \int\int_S \frac{d(\xi,\eta)}{r} d\xi d\eta \tag{44}$$

where

$$c_p = (K_1 + K_2)\pi, \; c_d = (L_2 - L_1)\pi \tag{45}$$

Substituting Eq. (30) into Eq. (44), we have

$$\delta - (Ax^2 + By^2) = \frac{(c_p p_0 + c_d d_0)}{a} \left\{ abK(e) - \frac{b}{a} D(e) x^2 - \frac{a}{b} [K(e) - D(e)] y^2 \right\} \tag{46}$$

where $K(e)$ and $D(e)$ are defined in Eq. (33). Comparing the coefficients on the left of Eq. (46) with those on the right, we have

$$\delta = (c_p p_0 + c_d d_0) bK(e) \tag{47}$$

$$A = (c_p p_0 + c_d d_0) \frac{b}{a^2} D(e) \tag{48}$$

$$B=(c_p p_0+c_d d_0)\frac{1}{b}[K(e)-D(e)] \tag{49}$$

Because the a, b, δ, p_0 and d_0 are all undetermined, following equilibrium equations are needed.

$$P_z=\iint_S p(x,y)dxdy=\frac{2}{3}\pi abp_0$$

$$Q=\iint_S d(x,y)dxdy=\frac{2}{3}\pi abd_0 \tag{50}$$

Using Eq. (48) and Eq. (49), we have

$$\frac{A}{B}=(1-e^2)\frac{D(e)}{K(e)-D(e)} \tag{51}$$

We can obtain e from Eq. (51), and then substituting $p_0=3P_z/(2\pi ab)$ and $d_0=3Q/(2\pi ab)$ into Eq. (47), Eq. (48) and Eq. (49) to obtain a, b and δ as follows:

$$a=n_a\left(\frac{c_pP_z+c_dQ}{\Sigma}\right)^{1/3},\ n_a=\left[\frac{3}{\pi}\left(1+\frac{B}{A}\right)D(e)\right]^{1/3}$$

$$b=n_b\left(\frac{c_pP_z+c_dQ}{\Sigma}\right)^{1/3},\ n_b=\left\{\frac{3}{\pi}\left(1+\frac{B}{A}\right)[K(e)-D(e)]\sqrt{1-e^2}\right\}^{1/3}$$

$$p_0=n_pP_z\left(\frac{\Sigma}{c_pP_z+c_dQ}\right)^{2/3},\ d_0=n_pQ\left(\frac{\Sigma}{c_pP_z+c_dQ}\right)^{2/3},\ n_p=\frac{3}{2\pi n_an_b}$$

$$\delta=n_\delta[(c_pP_z+c_dQ)^2\Sigma]^{1/3},\ n_\delta=\frac{3}{2\pi n_a}K(e) \tag{52}$$

where

$$\Sigma=2(A+B)=K_{11}+K_{12}+K_{21}+K_{22} \tag{53}$$

From the above equations, we can conclude

1. The expressions for n_a, n_b, n_p and n_δ are identical to those of classical contact mechanics and can be obtained if the value of A/B is determined.
2. From Eq. (51), we can see that if $e=0$, then $A/B=1$, and vice versa, so if $A=B$, the contact region is circle.
3. If $Q=0$, $P_z\neq 0$, then $d_0=0$, therefore there is no normal electric displacement in the contact region. If $P_z=0$, $Q\neq 0$, then $p_0=0$, therefore there is no normal pressure in the contact region and the deformation arises only from reverse-piezoelectric effect.

Obviously, the contact region for the contact between two spheres is circular, If we assume the radii of the two spheres are r_1 and r_2, then $K_{11}=K_{12}=1/r_1$, $K_{21}=K_{22}=1/r_2$, using Eq. (38) and Eq. (53), we have

$$A = B = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right),\ \Sigma = 2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \tag{54}$$

Substitution of Eq. (54) into Eq. (52) leads to

$$a = 0.9086\left[(c_p P_z + c_d Q)\frac{r_1 r_2}{r_1 + r_2}\right]^{1/3},\ p_0 = 0.5784 P_z\left[\frac{1}{c_p P_z + c_d Q}\frac{r_1 + r_2}{r_1 r_2}\right]^{2/3}$$

$$d_0 = 0.5784 Q\left[\frac{1}{c_p P_z + c_d Q}\frac{r_1 + r_2}{r_1 r_2}\right]^{2/3},\ \delta = 0.8255\left[(c_p P_z + c_d Q)^2\frac{r_1 + r_2}{r_1 r_2}\right]^{1/3} \tag{55}$$

where the numerical coefficients are obtained by calculating the n_a, n_b, n_p and n_δ in Eq. (52).

When a sphere with radius of r_1 contacts with a spherical concave surface with radius of r_2 ($r_2 > r_1$), we can obtain a, p_0, d_0 and δ by means of substituting r_2 in Eq. (55) for $-r_2$. In addition, when $r_2 \to \infty$, the surface of body 2 is a plane.

When two cylinders with the same radius r orthogonally contact with each other, the contact region is also circular since $K_{11} = K_{21} = 1/r$, $K_{12} = K_{22} = 0$ and $\omega = \pi/2$, so that with using of Eq. (38) and Eq. (53), we have

$$A = B = 1/(2r),\ \Sigma = 2/r \tag{56}$$

Substituting Eq. (56) into Eq. (52), we have

$$a = 0.9086[(c_p P_z + c_d Q)r]^{1/3},$$

$$p_0 = 0.5784 P_z\left[\frac{1}{(c_p P_z + c_d Q)r}\right]^{2/3}$$

$$d_0 = 0.5784 Q\left[\frac{1}{(c_p P_z + c_d Q)r}\right]^{2/3},$$

$$\delta = 0.8255\left[\frac{(c_p P_z + c_d Q)^2}{r}\right]^{1/3} \tag{57}$$

The a, p_0, d_0 and δ in Eq. (57) are same as those for the contact between a sphere with radius r and piezoelectric half-space.

Suppose that there is a PZT-4 piezoelectric half-space subjected to the action of a rigid sphere, a transversely isotropic sphere (assumed to have the same elastic constants as those of PZT-4) and a PZT-

Table 1
Material constants of PZT-4 and PZT-5H

Type of material	Elastic constants (10^{10}Nm^{-2})					Piezoelectric constants (Cm^{-2})			Dielectric constants (10^{10}CV^{-1}m^{-1}	
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	e_{31}	e_{33}	e_{15}	ε_{11}	ε_{33}
PZT-4	12.6	7.78	7.43	11.5	2.56	−5.2	15.1	12.7	64.9	56.2
PZT-5H	12.6	5.5	5.3	11.7	3.53	−6.5	23.3	17.0	151.	130.

5H sphere, respectively. The material constants of PZT-4 and PZT-5H are shown in Table 1. Based on the equations above, the values of contact parameter c_p and c_d in the above three cases are calculated and listed in Table 2 for comparison.

Based on these equations above, we could get the dimensions of the contact region, the values of the normal pressure $p(x,y)$ and the normal electric displacement $d(x,y)$ inside the contact region. In the next section, the corresponding elastic and electric fields in the piezoelectric sphere will be obtained by first evaluating the displacement functions and then differentiating. The displacement functions can be obtained by integrating the extended Boussinesq or Cerruti solutions in the contact region.

5. The elastic and electric fields for smooth contact between a piezoelectric sphere and another sphere under forces and charges

After getting the contact parameters in Section 4, we can now further solve the elastic and electric fields in the piezoelectric sphere.

5.1. Analytic solution

The contact stress and electric displacement inside the contact region are assumed as

$$\sigma_z(r,\theta) = -\frac{3P_z}{2\pi a^3}\sqrt{a^2 - r^2},\ D_z(r,\theta) = \frac{3Q}{2\pi a^3}\sqrt{a^2 - r^2},\ 0 \leq r \leq a \tag{58}$$

where the contact radius a is determined by Eq. (55),

Substituting $P_0 = -\sigma_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ and $Q_0 = D_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ into Eq. (28) and integrating the result over $0 \leq r_0 \leq a$, $0 \leq \theta_0 \leq 2\pi$, the displacement functions become

$$\psi_0(r,\theta,z) = 0$$

$$\psi_j(r,\theta,z) = \frac{3(P_z\delta_j + Q\lambda_j)}{2\pi a^3}\Pi(r,\theta,z_j),\ (j = 1,2,3) \tag{59}$$

where

$$\Pi(r,\theta,z_j) = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r_0^2}\ln R_j^* r_0 dr_0 d\theta_0 \tag{60}$$

The integral has been evaluated by Fabrikant (1988) with the result

Table 2
Contact parameter c_p and c_d

Type of sphere	c_p ($N^{-1}m^2$)	c_d ($C^{-1}m^2$)
Rigid sphere	9.1230×10^{-12}	-1.1100×10^{-2}
Transversely isotropic sphere	5.0476×10^{-11}	-1.1100×10^{-2}
PZT-5H sphere	1.7156×10^{-11}	-4.7126×10^{-3}

$$\Pi(r,\theta,z_j) = \frac{\pi}{2}\left\{ z_j\left(2a^2 - r^2 + \frac{2}{3}z_j^2\right)\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \frac{1}{3}\left[5r^2 - \frac{10}{3}a^2 - 2l_{2j}^2(a) - \frac{11}{3}l_{1j}^2(a)\right] + \frac{1}{3}\left[5r^2 - \frac{10}{3}a^2 - 2l_{2j}^2(a) - \frac{11}{3}l_{1j}^2(a)\right]\sqrt{a^2 - l_{1j}^2(a)} + \frac{4}{3}a^3\ln\left[l_{2j}(a) + \sqrt{l_{2j}^2(a) - r^2}\right]\right\} \tag{61}$$

where

$$l_{1j}(a) = \frac{1}{2}\left[\sqrt{(r+a)^2 + z_j^2} - \sqrt{(r-a)^2 + z_j^2}\right]$$

$$l_{2j}(a) = \frac{1}{2}\left[\sqrt{(r+a)^2 + z_j^2} + \sqrt{(r-a)^2 + z_j^2}\right] \tag{62}$$

Substituting Eq. (59) into Eq. (3), we obtain the elastic and electric fields as follows:

$$U = \frac{3}{2a^3}re^{i\theta}\sum_{j=1}^{3}(P_z\delta_j + Q\lambda_j)\left\{ -z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \sqrt{a^2 - l_{1j}^2(a)}\left[1 - \frac{l_{1j}^2(a) + 2a^2}{3r^2}\right] + \frac{2a^3}{3r^2}\right\}$$

$$w_m = \frac{3}{4a^3}\sum_{j=1}^{3}s_jk_{mj}(P_z\delta_j + Q\lambda_j)\left\{(2a^2 + 2z_j^2 - r^2)\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \frac{3l_{1j}^2(a) - 2a^2}{a}\sqrt{l_{2j}^2(a) - a^2}\right\}$$

$$\sigma_1 = \frac{6}{a^3}\sum_{j=1}^{3}(m_j - c_{66})(P_z\delta_j + Q\lambda_j)\left\{z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] - \sqrt{a^2 - l_{1j}^2(a)}\right\}$$

$$\sigma_2 = -\frac{2c_{66}}{a^3}\frac{e^{i2\theta}}{r^2}\sum_{j=1}^{3}(P_z\delta_j + Q\lambda_j)\left\{2a^3 - \left[l_{1j}^2(a) + 2a^2\right]\sqrt{a^2 - l_{1j}^2(a)}\right\}$$

$$\sigma_{zm} = \frac{3}{a^3}\sum_{j=1}^{3}\omega_{mj}(P_z\delta_j + Q\lambda_j)\left\{z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] - \sqrt{a^2 - l_{1j}^2(a)}\right\}$$

$$\tau_{zm} = \frac{3}{2a^3}re^{i\theta}\sum_{j=1}^{3}s_j\omega_{mj}(P_z\delta_j + Q\lambda_j)\left\{ -\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \frac{a\sqrt{l_{2j}^2(a) - a^2}}{l_{2j}^2(a)}\right\} \tag{63}$$

When the spheres are also subjected to tangential loading causing them to slide over each other, it is assumed that the sliding friction could be determined by the Coulomb friction law. Using the same method, the elastic and electric fields for sliding friction are obtained in Appendix A.

5.2. Numerical results

Suppose only the force P_z acts on a sphere ($Q = 0$) in contact with a half-space. Based on Eq. (63), the stresses in the half-space of PZT-4 ceramic are shown on the right side of Fig. 3 and Fig. 4. Meanwhile, based on the equations in Hanson (1992a), the stresses in the transversely isotropic half-space, which is assumed to have the same elastic constants as those of PZT-4, are shown on the left side of Fig. 3 and Fig. 4 for comparison. In addition, the electric potential and electric displacement are also shown on the right side of Fig. 3 and Fig. 4. The symbols in the figures are defined as

$$\tilde{\sigma}_z = \frac{\sigma_z}{p_m},\ \tilde{\sigma}_r = \frac{\sigma_r}{p_m},\ \tilde{\sigma}_\theta = \frac{\sigma_\theta}{p_m},\ \tilde{\tau}_1 = \frac{\tau_1}{p_m},\ p_m = \frac{P_z}{\pi a^2}$$

$$\tilde{\Phi} = \frac{\Phi}{P_z} \times 10^2,\ \tilde{D}_r = \frac{D_r}{P_z} \times 10^{10},\ \tilde{D}_z = \frac{D_z}{P_z} \times 10^{10} \tag{64}$$

where $\tau_1 = (\sigma_{\max} - \sigma_{\min})/2$ is the maximum shear stress at a point.

From the figures above, we can see

1. The greatest value of principal stress occurs at the center of the contact circle. Most points within the contact wide are under compression in three orthogonal directions.

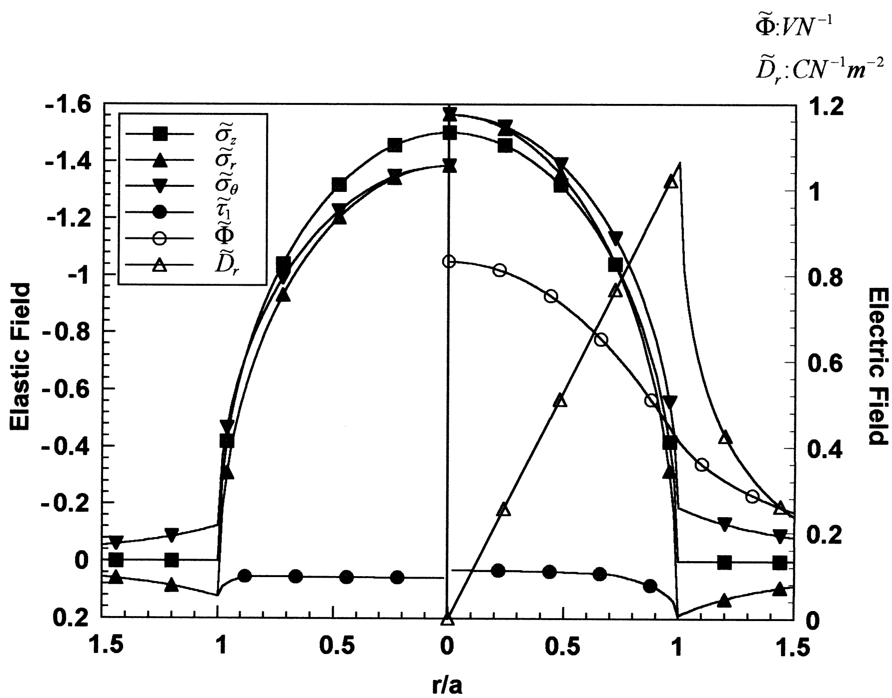

Fig. 3. Elastic and electric fields on surface.

2. For transversely isotropic media, the greatest value of the maximum shear stress is $\tau_1 = 0.4247 p_m$ and occurs on the axis of symmetry at the depth $z = 0.52a$. The value of the maximum shear stress on the circle $r = a$ on the surface is $\tau_1 = 0.1238 p_m$. For piezoelectric media, the greatest value of the maximum shear stress is $\tau_1 = 0.4467 p_m$ and also occurs on the axis of symmetry at the depth $z = 0.52a$. The value of the maximum shear stress on the circle $r = a$ on the surface is $\tau_1 = 0.1894 p_m$. Therefore, the cracks often occur on the axis of symmetry at the depth $z = 0.52a$ for either piezoelectric or non-piezoelectric transversely isotropic media. In addition, $\tau_1 = 0$ on the axis of symmetry at the depth $z = 0.02a$.
3. The greatest electric displacement D_z on the axis of symmetry occurs at the depth $z = 0.48a$. The electric displacement D_r on the surface rises linearly inside the contact circle, reaches the peak value on the contact circle $r = a$ and drops quickly outside the circle.

In addition, it should be noted that $D_z = 0$ on the surface and $D_r = 0$ on the axis of symmetry.

6. The elastic and electric fields for a smooth conical indentor on a transversely isotropic piezoelectric half-space under forces and charges

The problem under consideration is a cone-shaped indentor with vertex angle ϕ_0 pressed by a force P_z into a transversely isotropic piezoelectric half-space with planes of isotropy parallel to its surface. Meanwhile, a pair of charges $+Q$ and $-Q$ locate at two points on the axis of symmetry and in the indentor and half-space, respectively. The force and charges are all distant from the contact region.

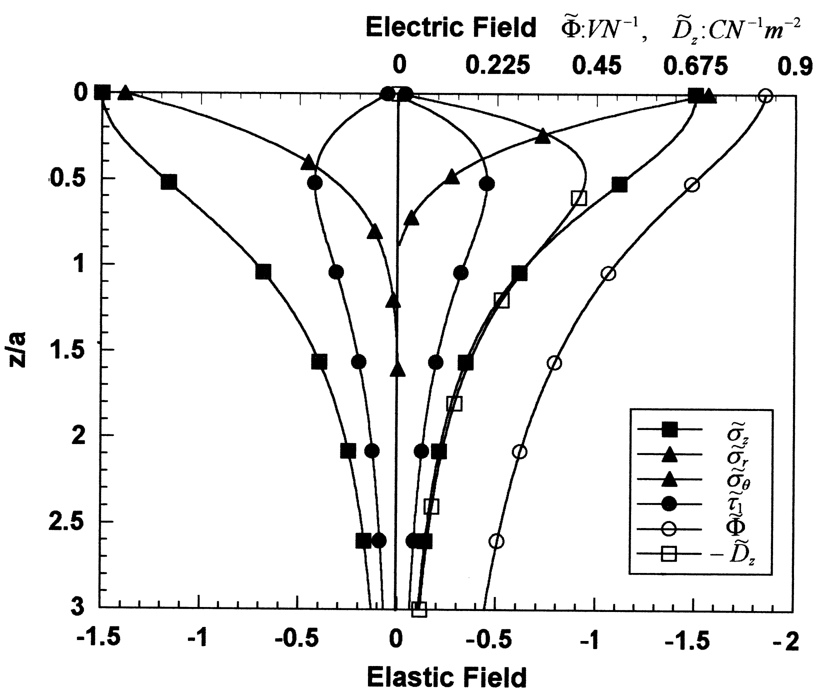

Fig. 4. Elastic and electric fields on axis of symmetry.

Suppose the surface of half-space coincides with the plane of isotropy, we can now solve the elastic and electric fields in the half-space.

6.1. Analytic solution

The contact stress and electric displacement inside the contact region are assumed as

$$\sigma_z(r,\theta) = -\frac{P_z}{\pi a^2}\cosh^{-1}\left(\frac{a}{r}\right), D_z(r,\theta) = \frac{Q}{\pi a^2}\cosh^{-1}\left(\frac{a}{r}\right), 0 \le r \le a \tag{65}$$

where a is the undetermined contact radius.

Substituting $P_0 = -\sigma_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ and $Q_0 = D_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ into Eq. (28) and integrating the result over $0 \le r_0 \le a$, $0 \le \theta_0 \le 2\pi$, the displacement functions become

$$\psi_0(r,\theta,z) = 0$$

$$\psi_j(r,\theta,z) = \frac{P_z\delta_j + Q\lambda_j}{\pi a^2}\Sigma(r,\theta,z_j), \ (j = 1,2,3) \tag{66}$$

where

$$\Sigma(r,\theta,z_j) = \int_0^{2\pi}\int_0^a \cosh^{-1}\left(\frac{a}{r_0}\right)\ln R_j^* r_0 dr_0 d\theta_0 \tag{67}$$

The integral has been evaluated by Hanson (1992b) with the result

$$\Sigma(r,\theta,z_j) = 2\pi\left\{ a z_j \sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \frac{1}{4}(2a^2 + r^2 - 2z_j^2)\ln\left[l_{2j}(a) + \sqrt{l_{2j}^2(a) - r^2}\right]\right.$$
$$\left. + \frac{3z_j l_{2j}(a)\left[r^2 - 2l_{1j}^2(a)\right]}{4r\sqrt{r^2 - l_{1j}^2(a)}} - \frac{3z_j\sqrt{r^2 + z_j^2}}{4} - \frac{1}{4}\left(r^2 - 2z_j^2\right)\ln\left(z_j + \sqrt{z_j^2 + r^2}\right)\right\} \tag{68}$$

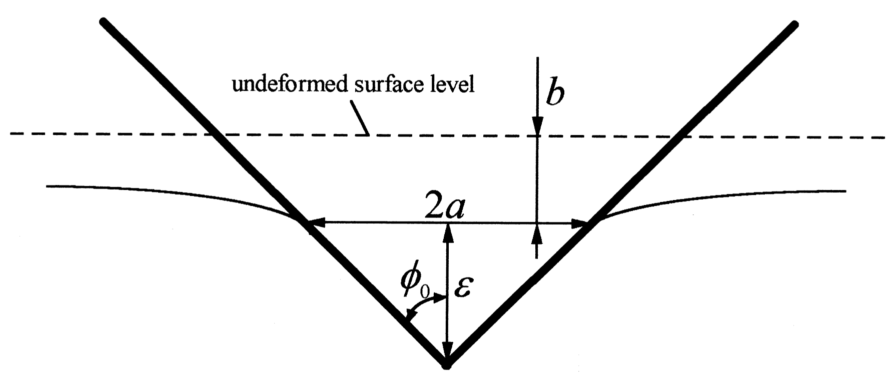

Fig. 5. Geometry of concial indentation.

We obtain the elastic and electric fields by substituting Eq. (66) into Eq. (3).

From Fig. 5, the normal displacement in the contact region can be written as

$$w = b + \varepsilon\left(1 - \frac{r}{a}\right), \ (0 \le r \le a) \tag{69}$$

where ε and ϕ_0 are related by $\varepsilon = a\cot\phi_0$. The normal displacement in the contact region ($z = 0$, $r \le a$) can also be expressed in the following form by using the elastic field obtained.

$$w = \frac{2H}{a^2}\left(\frac{\pi}{2}a - r\right) = \frac{2H}{a}\left(\frac{\pi}{2} - 1\right) + \frac{2H}{a}\left(1 - \frac{r}{a}\right) \tag{70}$$

where

$$H = \sum_{j=1}^{3} s_j k_{1j}(P_z\delta_j + Q\lambda_j) \tag{71}$$

Comparison of Eq. (69) and Eq. (70) yields

$$\varepsilon = \frac{2H}{a}, \ b = \varepsilon\left(\frac{\pi}{2} - 1\right) \tag{72}$$

If the force P_z and charges Q are specified, the contact parameters can be obtained as

$$a = \sqrt{2H\tan\phi_0}, \ \varepsilon = \sqrt{2H\cot\phi_0}, \ b = \varepsilon\left(\frac{\pi}{2} - 1\right) \tag{73}$$

When the conical indentor slides on the surface of half-space, it is assumed that the sliding friction could be determined by Coulomb friction law, and the elastic and electric fields for sliding friction are derived in Appendix B.

6.2. Numerical results

Suppose only force P_z acts on a conical indentor ($Q = 0$) contacting with a half-space. The stresses in the half-space of PZT-4 ceramic are shown on the right side of Fig. 6 and Fig. 7. Meanwhile, based on the equations in Hanson (1992b), the stresses in the transversely isotropic half-space, which is assumed to have the same elastic constants as those of PZT-4, are shown on the left side of Fig. 6 and Fig. 7 for comparison. In addition, the electric potential and electric displacement are also shown on the right side of Fig. 6 and Fig. 7. The symbols in the figures are defined in Eq. (64)

From the figures above, we can see

1. The principal stresses σ_r, σ_θ and σ_z are singular at the origin O. Most points within the contact wide are under compression in three orthogonal directions.
2. For transversely isotropic media, the greatest value of the maximum shear stress occurs near the origin o. The value of the maximum shear stress on the axis of symmetry is $\tau_1 = 0.8901p_m$ at the depth $z = 0.02a$, which is much bigger than the $\tau_1 = 0.1238p_m$ on the contact circle $r = a$. For piezoelectric media, the greatest value of the maximum shear stress on the axis of symmetry is $\tau_1 = 0.7889p_m$ at the depth $z = 0.1a$, which is bigger than the $\tau_1 = 0.1894p_m$ on the contact circle $r = a$.
3. The greatest electric displacement D_z on the axis of symmetry occurs near the origin O. The electric

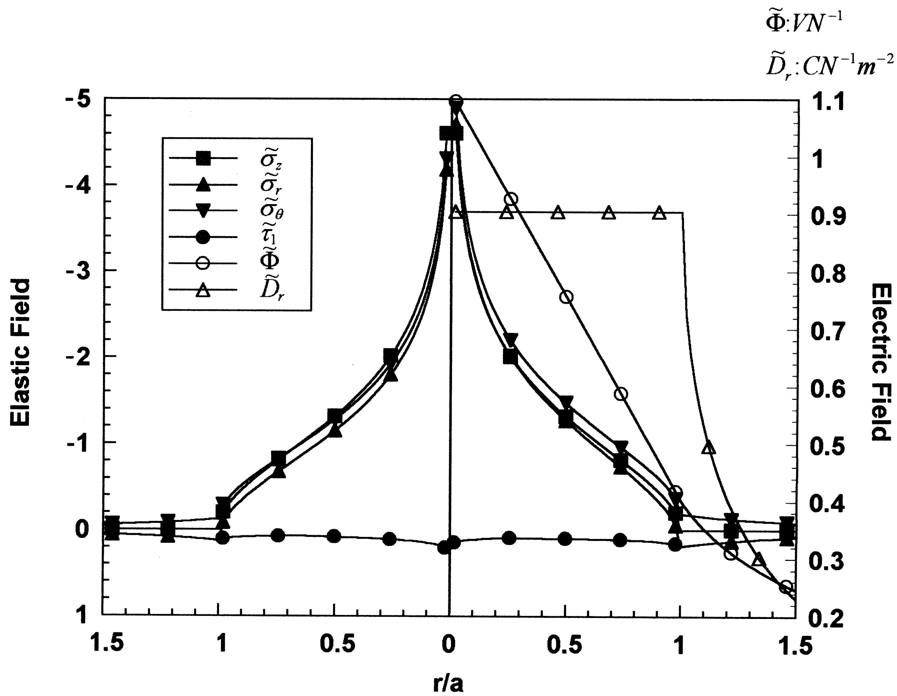

Fig. 6. Elastic and electric fields in surface.

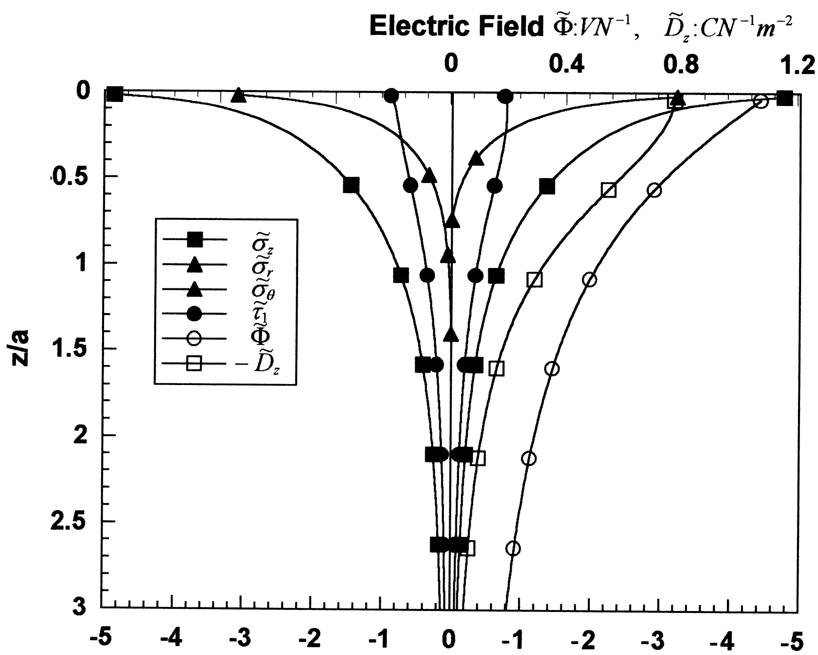

Fig. 7. Elastic and electric fields on axis of symmetry.

displacement D_r on the surface is a constant value inside the contact circle $r = a$ and drops rapidly outside the circle.

As for spherical contact, $D_z = 0$ on the surface and $D_r = 0$ on the axis of symmetry.

7. The elastic and electric fields for a smooth upright circular flat punch on a transversely isotropic piezoelectric half-space under forces and charges

The problem under consideration is of a circular flat punch with radius a pressed into a transversely isotropic piezoelectric half-space by force P_z. Meanwhile, a pair of charges $+Q$ and $-Q$ located at two points on the axis of symmetry and in the punch and half-space, respectively. The force and charges are all distant from the contact region. Suppose the surface of half-space coincides with the plane of isotropy, we can now solve the elastic and electric fields in the half-space.

7.1. Analytic solution

The contact stress and electric displacement inside the contact region are assumed as

$$\sigma_z(r,\theta) = -\frac{P_z}{2\pi a}(a^2 - r^2)^{-1/2},\ D_z(r,\theta) = \frac{Q}{2\pi a}(a^2 - r^2)^{-1/2},\ 0 \le r \le a \tag{74}$$

Substituting $P_0 = -\sigma_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ and $Q_0 = D_z(r_0,\theta_0)r_0 dr_0 d\theta_0$ into Eq. (28) and integrating the result over $0 \le r_0 \le a$, $0 \le \theta_0 \le 2\pi$, the displacement functions become

$$\psi_0(r,\theta,z) = 0$$

$$\psi_j(r,\theta,z) = \frac{P_z\delta_j + Q\lambda_j}{2\pi a}\Theta(r,\theta,z_j),\ (j = 1,2,3) \tag{75}$$

where

$$\Theta(r,\theta,z_j) = \int_0^{2\pi}\int_0^a (a^2 - r_0^2)^{-1/2}\ln R_j^* r_0 dr_0 d\theta_0 \tag{76}$$

The integral has been evaluated by Fabrikant (1988) with the result

$$\Theta(r,\theta,z_j) = 2\pi\left\{z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] - \sqrt{a^2 - l_{1j}^2(a)} + a\ln\left[l_{2j}(a) + \sqrt{l_{2j}^2(a) - r^2}\right]\right\} \tag{77}$$

We obtain the elastic and electric fields by substituting Eq. (75) into Eq. (3).

When upright circular flat punch slides on the surface of half-space, it is assumed that the sliding friction could be determined by Coulomb friction law, and the elastic and electric fields for sliding friction are derived in Appendix C.

7.2. Numerical results

Suppose only force P_z acts on a circular flat punch ($Q = 0$) contacting with half-space. The stresses in the half-space of PZT-4 ceramic are shown on the right side of Fig. 8 and Fig. 9. Meanwhile, based on the equations in Fabrikant (1988), the stresses in the transversely isotropic half-space, which is assumed

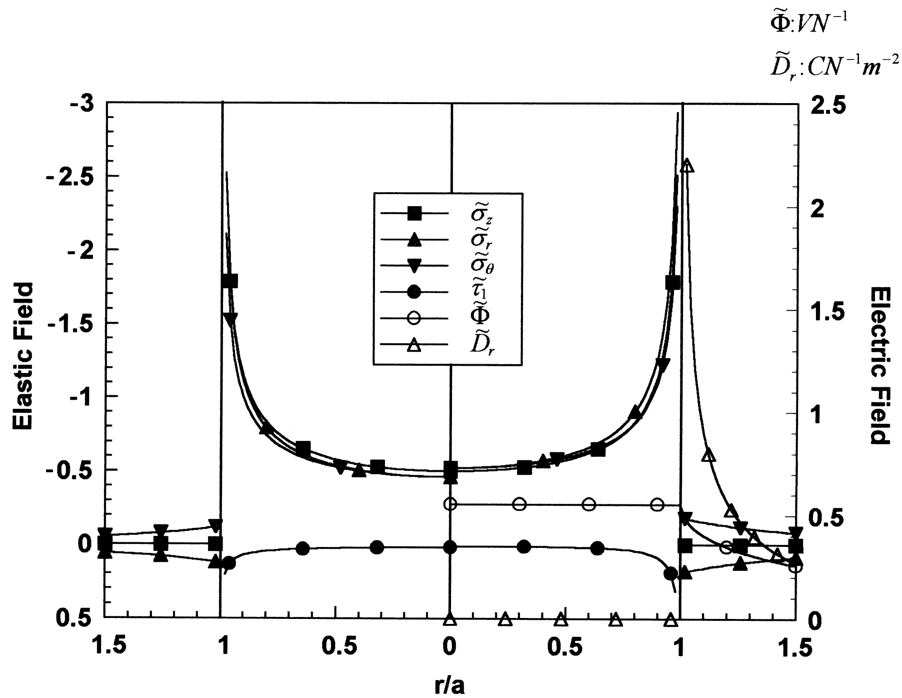

Fig. 8. Elastic and electric fields on surface.

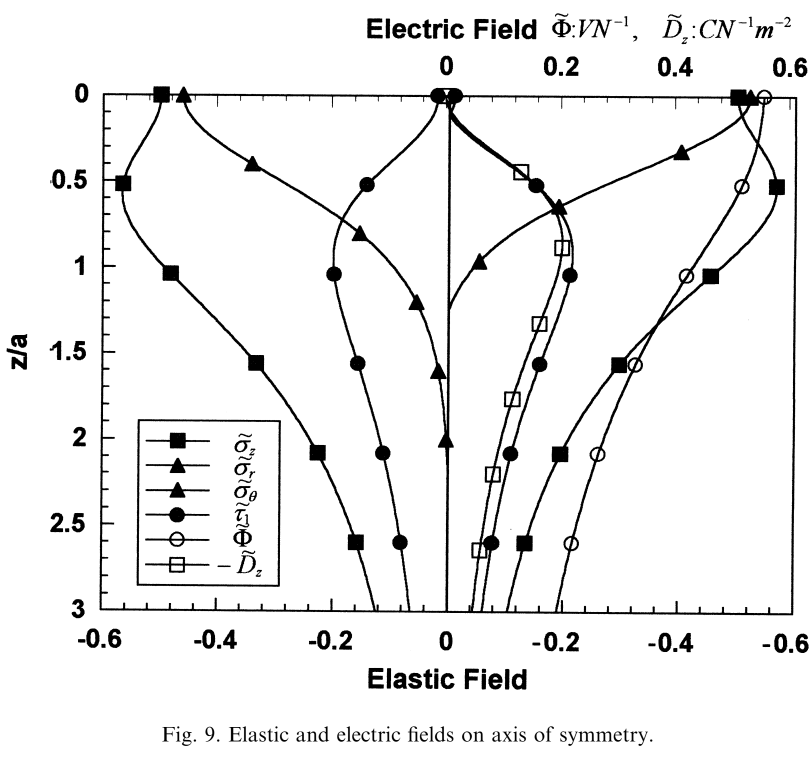

Fig. 9. Elastic and electric fields on axis of symmetry.

to have the same elastic constants as those of PZT-4, are shown on the left side of Fig. 8 and Fig. 9 for comparison. In addition, the electric potential and electric displacement are also shown on the right side of Fig. 8 and Fig. 9. The symbols in the figures are defined in Eq. (64).

From the figures above, we can see

1. The principal stresses σ_r, σ_θ and σ_z are singular on the contact circle $r=a$, and the points inside the contact circle are all under compression in three orthogonal directions.
2. For transversely isotropic media, the greatest value of the maximum shear stress occurs near the contact circle $r=a$, with the $\tau_1=0.2078p_m$ on the surface circle $r = 0.98a$. The greatest value of the maximum shear stress on the axis of symmetry is $\tau_1=0.2002p_m$ at the depth $z = 0.96a$. For piezoelectric media, the greatest value of the maximum shear stress also occurs near the contact circle $r=a$, with the $\tau_1=0.3180p_m$ on the surface circle $r = 0.98a$. The greatest value of the maximum shear stress on the axis of symmetry is $\tau_1=0.2143p_m$ at the depth $z = 0.92a$. In addition, $\tau_1=0$ on the axis of symmetry at the depth $z = 0.1a$.
3. The greatest electric displacement D_z on the axis of symmetry occurs at the depth $z = 0.84a$. The electric displacement D_r on the surface is zero inside the contact circle and singular on the contact circle, and then it drops rapidly outside the circle.

As for spherical contact, $D_z=0$ on the surface and $D_r=0$ on the axis of symmetry.

Acknowledgements

This work is supported by the National Natural Science Foundation of China and the Zhejiang Provincial Natural Science Foundation and the Research Laboratory of Failure Mechanics of the State Education Commission of China.

Appendix A. The elastic and electric fields for sliding friction between a piezoelectric sphere and another sphere

According to the Coulomb friction law, the sliding friction in the contact region is taken as a coefficient of friction multiplied by the contact pressure. Using f_x and f_y as the coefficients of friction in the x and y directions, the complex shear force $P=P_x+iP_y$ in the displacement functions of Eq. (28) is replaced by

$$P = \frac{3P_z f}{2\pi a^3}\sqrt{a^2 - r_0^2}\, r_0 dr_0 d\theta_0, f = f_x + if_y \tag{A1}$$

and the result is integrated over $0 \le r_0 \le a$, $0 \le \theta_0 \le 2\pi$. The displacement functions become

$$\psi_0(r,\theta,z) = i\frac{3P_zG_0}{2\pi a^3}(f\bar{\Delta} - \bar{f}\Delta)[z_0\Pi(r,\theta,z_0) - \Omega(r,\theta,z_0)]$$

$$\psi_j(r,\theta,z) = \frac{3P_zG_j}{2\pi a^3}(f\bar{\Delta} + \bar{f}\Delta)[z_j\Pi(r,\theta,z_j) - \Omega(r,\theta,z_j)] \tag{A2}$$

where $\Pi(r,\theta,z_j)$ is given in Eq. (61), and $\Omega(r,\theta,z_j)$ is

$$\Omega(r,\theta,z_j) = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r_0^2}R_j r_0 dr_0 d\theta_0 \tag{A3}$$

The integral has been evaluated by Hanson (1992a) with the result

$$\Omega(r,\theta,z_j) = \frac{\pi}{32}\left\{\left[8\left(a^2+z_j^2\right)^2+8a^2r^2-r^4-8r^2z_j^2\right]\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\left[2a^2\left(4a^2+r^2-4z_j^2\right)\right.\right.$$
$$\left.\left.+\,l_{1j}^2(a)\left(6a^2-r^2+14z_j^2\right)\right]\frac{z_j}{\sqrt{a^2-l_{1j}^2(a)}}\right\} \tag{A4}$$

Substituting Eq. (A2) into Eq. (3), we obtain the elastic and electric fields as follows:

$$U = \frac{3P_z}{2a^3}\sum_{j=1}^{3}G_j\left[f\left\{\left(\frac{1}{2}r^2-a^2-z_j^2\right)\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\frac{2a^2-3l_{1j}^2(a)}{2l_{1j}(a)}\sqrt{r^2-l_{1j}^2(a)}\right\}\right.$$

$$\left.+\bar{f}e^{i2\theta}\left\{-\frac{4a^3z_j}{3r^2}+\frac{1}{4}r^2\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\left[8a^4+a^2r^2-l_{1j}^2(a)\times\left(z_j^2+\frac{5}{2}r^2+5a^2\right)\right]\frac{\sqrt{r^2-l_{1j}^2(a)}}{6r^2l_{1j}(a)}\right\}\right]$$

$$-\frac{3P_zG_0}{2a^3}\left[f\left\{\left(\frac{1}{2}r^2-a^2-z_0^2\right)\sin^{-1}\left[\frac{l_{10}(a)}{r}\right]+\frac{2a^2-3l_{10}^2(a)}{2l_{10}(a)}\sqrt{r^2-l_{10}^2(a)}\right\}\right.$$

$$\left.-\bar{f}e^{i2\theta}\left\{-\frac{4a^3z_0}{3r^2}+\frac{1}{4}r^2\sin^{-1}\left[\frac{l_{10}(a)}{r}\right]+\left[8a^4+a^2r^2-l_{10}^2(a)\times\left(z_0^2+\frac{5}{2}r^2+5a^2\right)\right]\frac{\sqrt{r^2-l_{10}^2(a)}}{6r^2l_{10}(a)}\right\}\right]$$

$$w_m = \frac{3P_z}{2a^3}(\bar{f}e^{i\theta}+fe^{-i\theta})r\sum_{i=1}^{3}s_jk_{mj}G_j\left\{-z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\sqrt{a^2-l_{1j}^2(a)}\left[1-\frac{l_{1j}^2(a)+2a^2}{3r^2}\right]+\frac{2a^3}{3r^2}\right\}$$

$$\sigma_1 = \frac{3P_z}{a^3}(\bar{f}e^{i\theta}+fe^{-i\theta})r\sum_{j=1}^{3}(m_j-c_{66})G_j\left\{-\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\sqrt{r^2-l_{1j}^2(a)}\frac{l_{1j}(a)}{r^2}\right\}$$

$$\sigma_2 = \frac{6c_{66}P_z}{a^3}\sum_{j=1}^{3}G_j\left[fe^{i\theta}\left\{\frac{1}{2}r\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]-\frac{l_{1j}(a)}{2r}\sqrt{r^2-l_{1j}^2(a)}\right\}\right.$$

$$\left.-\bar{f}e^{i3\theta}\left\{4\left(\left[l_{1j}^2(a)+2a^2\right]\sqrt{a^2-l_{1j}^2(a)}-2a^3\right)\frac{z_j}{3r^3}+\frac{l_{1j}^3(a)}{r^3}\sqrt{r^2-l_{1j}^2(a)}\right\}\right]$$

$$-\frac{6c_{66}P_zG_0}{a^3}\left[fe^{i\theta}\left\{\frac{1}{2}r\sin^{-1}\left[\frac{l_{10}(a)}{r}\right]-\frac{l_{10}(a)}{2r}\sqrt{r^2-l_{10}^2(a)}\right\}\right.$$

$$\left.+\bar{f}e^{i3\theta}\left\{4\left(\left[l_{10}^2(a)+2a^2\right]\sqrt{a^2-l_{10]}^2(a)}-2a^3\right)\frac{z_0}{3r^3}+\frac{l_{10}^3(a)}{r^3}\sqrt{r^2-l_{10}^2(a)}\right\}\right]$$

$$\sigma_{zm}=\frac{3P_z}{2a^3}(\bar{f}e^{i\theta}+fe^{-i\theta})r\sum_{j=1}^{3}G_j\omega_{mj}\left\{-\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\frac{l_{1j}(a)}{r^2}\sqrt{r^2-l_{1j}^2(a)}\right\}$$

$$\tau_{zm}=\frac{3P_z}{a^3}\sum_{j=1}^{3}s_j\omega_{mj}G_j\left[f\left\{-z_j\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right]+\sqrt{a^2-l_{1j}^2(a)}\right\}\right.$$

$$\left.-\bar{f}e^{i2\theta}\frac{2a^3-\left[2a^2+l_{1j}^2(a)\right]\sqrt{a^2-l_{1j}^2(a)}}{3r^2}\right]$$

$$-\frac{3\rho_m s_0G_0P_z}{a^3}\left[f\left\{-z_0\sin^{-1}\left[\frac{l_{10}(a)}{r}\right]+\sqrt{a^2-l_{10}^2(a)}\right\}\right.$$

$$\left.+\bar{f}e^{i2\theta}\frac{2a^3-\left[2a^2+l_{10}^2(a)\right]\sqrt{a^2-l_{10}^2(a)}}{3r^2}\right] \tag{A5}$$

We want to note that the above solution for shear loading are approximate since the tangential displacement will not generally align with the shear traction. The tangential traction will also produce a normal displacement in the contact region thus altering the contact pressure. To consider these details further, the surface displacement in the contact region will be examined.

From Eq. (A5), we can get

$$U=\frac{3\pi P_zf}{4a^3}\alpha\left(a^2-\frac{r^2}{2}\right)+\frac{3P_z\bar{f}}{2a^3}\beta e^{i2\theta}\left[\left(\frac{\pi}{8}+a^2\right)r^2+8a^4\right]$$

$$w=\frac{P_z}{a^3}\gamma(\bar{f}e^{i\theta}+fe^{-i\theta})\frac{a^3-(a^2-r^2)^{3/2}}{r},\ r\le a \tag{A6}$$

where

$$\alpha=G_0-\sum_{j=1}^{3}G_j,\ \beta=\sum_{j=0}^{3}G_j,\ \gamma=\sum_{j=1}^{3}s_jk_{1j}G_j \tag{A7}$$

It is apparent in this case that the tangential displacements do not align with the shear traction. This results from the coupling of the second term in the first equation of Eq. (A6). In the special case $\beta=0$ the coupling vanishes and the displacements and traction align. The normal displacement is also nonzero except when $\gamma=0$. So only when $\beta=0$ and $\gamma=0$ are satisfied simultaneously, the solution derived here for sliding sphere is an exact solution.

Appendix B. The elastic and electric fields for sliding friction of a conical indentor on a transversely isotropic piezoelectric half-space

According to the Coulomb friction law, the complex shear force $P=P_x+iP_y$ in the displacement functions of Eq. (28) is replaced by

$$P=\frac{P_z f}{\pi a^2}\cosh^{-1}\left(\frac{a}{r_0}\right)r_0 dr_0 d\theta_{0_y},\ f=f_x+if_y \tag{B1}$$

and the result is integrated over $0\le r_0\le a$, $0\le\theta_0\le 2\pi$. The displacement functions become

$$\psi_0(r,\theta,z)=i\frac{P_zG_0}{\pi a^2}(f\bar{\varDelta}-\bar{f}\varDelta)[z_0\Sigma(r,\theta,z_0)-\Gamma(r,\theta,z_0)]$$

$$\psi_j(r,\theta,z)=\frac{P_zG_j}{\pi a^2}(f\bar{\varDelta}+\bar{f}\varDelta)[z_j\Sigma(r,\theta,z_j)-\Gamma(r,\theta,z_j)] \tag{B2}$$

where $\Sigma(r,\theta,z_j)$ is given in Eq. (68), and $\Gamma(r,\theta,z_j)$ is

$$\Gamma(r,\theta,z_j)=\int_0^{2\pi}\int_0^a\cosh^{-1}\left(\frac{a}{r_0}\right)R_jr_0dr_0d\theta_0 \tag{B3}$$

The integral has been evaluated by Hanson (1992b) with the result

$$\Gamma(r,\theta,z_j)=2\pi\left\{\frac{a}{12}\left(2a^2+6z_j^2+3r^2\right)\sin^{-1}\left[\frac{1_{1j}(a)}{r}\right]+\frac{1}{3}z_j^3\ln\left(z_j+\sqrt{z_j^2+r^2}\right)\right.$$

$$-\frac{1}{3}z_j^3\ln\left[l_{2j}(a)+\sqrt{l_{2j}^2(a)-r^2}\right]-\frac{1}{9}\left(4z_j^2+r^2\right)\sqrt{z_j^2+r^2}$$

$$\left.+\frac{1}{36}\left[21l_{1j}^2(a)+16l_{2j}^2(a)-12r^2-10a^2\right]\sqrt{l_{2j}^2(a)-a^2}\right\} \tag{B4}$$

We obtain the elastic and electric fields by substituting Eq. (B2) into Eq. (3).

The elastic and electric fields obtained are approximate for the same reasons as in Appendix A. We consider these details further by examining the surface displacement as follows:

$$U=\frac{2P_zf}{a^2}\alpha\left(\frac{\pi a}{2}-r\right)+\frac{2P_z\bar{f}}{3a^2}\beta re^{i2\theta}$$

$$w=\frac{P_z}{a^2}\gamma(\bar{f}e^{i\theta}+fe^{-i\theta})\left[r\ln\frac{a+\sqrt{a^2-r^2}}{r}+\frac{a\left(a-\sqrt{a^2-r^2}\right)}{r}\right],\ r\le a \tag{B5}$$

where α, β and γ are defined in Eq. (A7).

It is apparent in this case that the tangential displacements do not align with the shear traction. This results from the coupling of the second term in the first equation of Eq. (B5). In the special case $\beta=0$ the coupling vanishes and the displacements and traction align. The normal displacement is maximum near the edge of contact at $r=0.827a$ and vanishes at the center $r=0$. It is also nonzero except when

$\gamma = 0$. So the solution derived here for sliding conical indentor is an exact solution only for materials with $\beta = 0$ and $\gamma = 0$, just like the solution for the sliding sphere.

Appendix C. The elastic and electric fields for sliding friction of an upright circular flat punch on a transversely isotropic piezoelectric half-space

According to the Coulomb friction law, the complex shear force $P = P_x + iP_y$ in the displacement functions of Eq. (28) is replaced by

$$P = \frac{P_z f}{2\pi a}\left(a^2 - r_0^2\right)^{1/2} r_0 dr_0 d\theta_0, f = f_x + if_y \tag{C1}$$

and the result is integrated over $0 \le r_0 \le a$, $0 \le \theta_0 \le 2\pi$. The displacement functions become

$$\psi_0(r,\theta,z) = i\frac{P_z G_0}{2\pi a}(f\bar{\Delta} - \bar{f}\Delta)[z_0\Theta(r,\theta,z_0) - \Theta(r,\theta,z_0)]$$

$$\psi_j(r,\theta,z) = \frac{P_z G_j}{2\pi a}(f\bar{\Delta} + \bar{f}\Delta)[z_j\Theta(r,\theta,z_j) - \Theta(r,\theta,z_j)] \tag{C2}$$

where $\Theta(r,\theta,z_j)$ is given in Eq. (77), and $\Theta(r,\theta,z_j)$ is

$$\Theta(r,\theta,z_j) = \int_0^{2\pi}\int_0^a \left(a^2 - r_0^2\right)^{-1/2} R_j r_0 dr_0 d\theta_0 \tag{C3}$$

The integral has been evaluated by Hanson (1994) with the result

$$\Theta(r,\theta,z_j) = \frac{\pi}{2}\left\{\left(2a^2 + 2z_j^2 + r^2\right)\sin^{-1}\left[\frac{l_{1j}(a)}{r}\right] + \frac{1}{a}\left[2a^2 + l_{1j}^2(a)\right]\sqrt{l_{2j}^2(a) - a^2}\right\} \tag{C4}$$

We obtain the elastic and electric fields by substituting Eq. (C2) into Eq. (3).

Similarly, we further examine the surface displacement in the contact region as follows:

$$U = \frac{\pi P_z f}{2a}\alpha, w = \frac{P_z}{a^2}\gamma(\bar{f}e^{i\theta} + fe^{-i\theta})\left(\frac{a - \sqrt{a^2 - r^2}}{r}\right), r \le a \tag{C5}$$

where α and γ are defined in Eq. (A7).

From the first equation of Eq. (C5) it is apparent that f is the only complex quantity on the right hand side and therefore one can write $U = u + iv = C(f_x + if_y)$ where C is a real constant. In this case the tangential displacement does align with the direction of shear traction and the solution is exact. The second equation of Eq. (C5) reveals that the normal displacement in the contact region is nonzero and this will alter the contact pressure. It has a maximum value at $r = a$ and vanishes at $r = 0$.In the special case of $\gamma = 0$ the normal displacement vanishes everywhere on the surface and there is no interaction. It may now be concluded that the solution derived here for a sliding upright punch is an exact solution for materials with $\gamma = 0$. For $\gamma \neq 0$ the shear stress and tangential displacement are still in alignment but the solution is not exact since the shear loading will alter the contact pressure.

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